ON OPTIMAL ADVERTISING CAPITAL AND RESEARCH EXPENDITURES UNDER DYNAMIC CONDITIONS

BY
PHOEBUS J. DHRYMES

TECHNICAL REPORT NO. 112
APRIL 13, 1962

PREPARED UNDER CONTRACT Nonr-225(50)
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FOR
OFFICE OF NAVAL RESEARCH

INSTITUTE FOR MATHEMATICAL STUDIES IN THE SOCIAL SCIENCES
Applied Mathematics and Statistics Laboratories
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Stanford University

0. Preliminaries:

The purpose of this paper is to extend the results obtained by Nerlove and Arrow in a recent paper. In particular, it will be shown that an analysis similar to theirs can be employed to study the characterization of optimal expenditures policies where the firm is permitted to manipulate not only its demand but its cost function as well.

We deal first with the case where the only additional policy variable employed is expenditure on the capital stock. We shall employ assumptions parallel to those of Nerlove and Arrow [1].

1. Formulation of the Problem.

Consider a firm with demand function

$$q = f(p,\ A_1,\ z)\ , \quad f \in C'' \tag{1}$$

where p is price of output, A_1 is level of "good will" and z is an exogenous variable.

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The firm's cost function is given by:

$$(2) \qquad C = C(q, A_2) \quad C \in C''$$

where q is output and A_2 is the capital stock.

It is assumed, if necessary after suitable normalization, that a dollar's expenditure on the capital stock increments it by one unit, and similarly with respect to advertising expenditure and the level of "good will." If both stocks are assumed to decay exponentially, then we have the relations

$$(3) \qquad \begin{aligned} \dot{A}_1 &= a_1 - \delta_1 A_1 \\ \dot{A}_2 &= a_2 - \delta_2 A_2 \end{aligned}$$

where the a_i denote expenditure on the i^{th} stock and δ_i are positive constants. The initial stocks are given by:

$$(3a) \qquad \begin{aligned} A_1(0) &= A_{10} \\ A_2(0) &= A_{20} \, . \end{aligned}$$

The problem now is to choose $p(t)$, $A_1(t)$, $A_2(t)$ so as to maximize

$$(4) \qquad V(A_1,A_2,p,z) = \int_0^\infty e^{-\alpha t} \left[pf(p,A_1,z) - C(q,A_2) - \sum_{i=1}^{2} a_i \right] dt - \sum_{i=1}^{2} A_{io}$$

subject to (3) and (3a). The inclusion of the term $\sum_{i=1}^{2} A_{io}$ in the maximand of (4) is innocuous since it is a fixed number. If holding A_1, A_2, z constant we maximize (4) with respect to p, we shall obtain a solution

$$(4a) \qquad \hat{p}(t) = P(A_1, A_2, z) \, ,$$

which gives the optimal time path in terms of A_1, A_2, and z.

Inserting (4) and (3) into (5) and performing an integration by parts yields:

$$S(A_1, A_2, z) = \int_0^\infty e^{-\alpha t} \left[R(A_1, A_2, z) - \sum_{i=1}^{2} (\alpha + \delta_i) A_i \right] dt \tag{5}$$

where $R(A_1, A_2, z) = \hat{p}\, f(p, A_1, z) - C(\hat{q}, A_2)$ and we have assumed

$$\lim_{t \to \infty} e^{-\alpha t} [A_1(t) + A_2(t)] = 0 \tag{5a}$$

for the optimal $A_1(t)$, $A_2(t)$. As will be shown later, this involves little loss of generality.

From (5) we must now deduce the characterization of the optimal policies $A_1^*(t)$, $A_2^*(t)$.

2. Characterization of the Solution.

Before we proceed it is necessary to impose some conditions on the function $R(A_1, A_2, z)$, for otherwise it is not possible to say anything about the solution. More specifically,

Assumption 1. For given z, the function $R(A_1, A_2, z)$ is continuous and bounded in A_1, A_2. If z is interpreted as national income, boundedness is indeed a required specification on R since, if not imposed, we could have the profits of this firm exceeding national income.

We have, for given z:

Lemma 2.1. There exist policies $A_i(t)$, $i = 1, 2$, maximizing (5) and they are finite.

Proof: Let $\Pi(A_1,A_2,z) = R(A_1,A_2,z) - \sum_{i=1}^{2} (\alpha+\delta_i)A_i$. Now if, for every t, Π is maximized for given z, then so is (5). On the other hand, if (5) is maximized then $\Pi(A_1,A_2,z)$ must be maximized for each interval of t. Hence we shall merely show that Π assumes its maximum on the finite plane.

Since $R(A_1,A_2,z)$ is bounded, it follows that there exists a pair $(\bar{A}_1,\bar{A}_2)$ such that $\Pi(A_1,A_2,z)$ is decreasing for $A_1 > \bar{A}_1$ $A_2 > \bar{A}_2$. Thus we need consider Π on the rectangle $0 \leq A_1 \leq \bar{A}_1$, $0 \leq A_2 \leq \bar{A}_2$. Since Π is continuous there exists a pair (A_1^*,A_2^*) such that $\Pi(A_1^*,A_2^*,z) = \max.$ for given z. It is, of course, to be understood that in general A_1^*, A_2^* are functions of z.

Remark 2.2. This shows that the assumption (5a) does not restrict the generality of our result, since the lemma states that for bounded z optimal policies are finite, hence that $e^{\alpha t}$ grows "faster" than they do.

Assumption 2. The function $R(A_1,A_2\, z)$ is strictly concave in A_1,A_2 for given z.

Lemma 2.3. The maximum of Lemma 2.1 is absolute and the functions $A_1^*(z)$, $A_2^*(z)$ are continuous in z.

Proof: The first part of the lemma is redundant: we give the proof here in the interest of completeness.

Thus let $X = (A_1,A_2)$, D the domain of Π. Then for any $Y \epsilon D$ and sufficiently small λ,

$$\prod(X^*) \geq \prod[(1-\lambda)\, X^* + \lambda Y] \geq (1-\lambda)\prod(X^*) + \lambda \prod(Y)$$

or

$$\prod(X^*) \geq \prod(Y) .$$

For the second part we argue as follows: Let $\{z_i\colon\ i = 1, 2, \ldots\}$ be a sequence such that $\lim_{i \to \infty} z_i = z_o$. This defines the sequence $\{X^*(z_i) = (A_1^*(z_i), A_2^*(z_i))\ i = 1, 2, \ldots\}$. Suppose $\lim_{i \to \infty} X^*(z_i) = X \neq X^*(z_o)$. Then since the $X^*(z_i)$ represent maximizing policies we find for sufficiently small λ

$$\prod(X) \geq \prod[(1-\lambda)\, X + \lambda X^*(z_o)] \geq (1-\lambda)\prod[X] + \lambda \prod[X^*(z_o)]$$

or

$$\prod(X) \geq \prod[X^*(z_o)] .$$

But strict inequality is impossible since $X^*(z_o)$ is the maximizing set for z_o.

Thus the $A_1^*(z)$, $A_2^*(z)$ are continuous in z.

Lemma 2.4. If (A_1^*, A_2^*) is the maximum for $\prod(A_1, A_2)$ the optimum expenditure policy for $A_{io} \leq A_i^*$, $i = 1, 2$, $t > 0$, is given by

$$a_i^* = \delta_i A_i^* \qquad i = 1, 2$$

and

$$\begin{aligned} a_i^* &= \delta_i A_i^* \quad \text{if} \quad A_{io} = A_i^* \qquad i = 1, 2,\ t = 0 \\ &= +\infty \qquad\qquad A_{io} < A_i^* . \end{aligned}$$

Proof: Since expenditures are not constrained, the assertion for $A_{io} < A_i^*$, $t = 0$, follows from the fact that (A_1^*, A_2^*) is a maximum.

The assertion for $t > 0$, $A_{1o} = A_1^*$, is an immediate consequence of (3) and Lemma 2.3.

It remains next to deal with the situation where $A_{io} > A_i^*$ for at least one i. Now the decay of stocks constrains the manner of decumulation, since stocks cannot be decumulated faster than they decay. Let $\mathcal{P}$ be the admissible family of decumulation paths. For any path $\Gamma \in \mathcal{P}$, let $c(\Gamma)$ be the cost associated with it; note that an admissible path may involve partial maintenance for an otherwise redundant stock.

The problem is then to find a path Γ^* minimizing the line integral

$$\int_\Gamma [\Pi(A_1^*, A_2^*, z) - \Pi(A_1, A_2, z)] - c(\Gamma) \tag{6}$$

with the constraint $\Gamma \in \mathcal{P}$. This is itself a calculus of variations problem which is too cumbersome to be considered here. Instead we shall show that, by restricting the form of the function Π, we obtain a simple characterization of the solution affording maximal correspondence with the results of Nerlove and Arrow.

Observe first that for given $\bar{A}_1$, $\bar{A}_2$, the functions $\Pi(\bar{A}_1, A_2)$, $\Pi(A_1, \bar{A}_2)$ are concave in A_2, A_1, respectively. To this we add:

Assumption 3. The function $\Pi(A_1, A_2)$ has a maximum at A_2^* for any fixed A_1, and a maximum at A_1^* for any fixed A_2.

It should be pointed out that while Assumptions 1 and 2 have their direct analogues in [1], Assumption 3 deals with a situation that does not arise in the problem of Nerlove and Arrow.

We have:

Lemma 2.5. If $A_{1o} > A_i^*$ $i = 1, 2$ then there exist constants τ_i $i = 1, 2$ such that optimal expenditure policies are given by:

$$a_i^* = 0 \qquad \text{for} \qquad o \le t \le \tau_i \qquad i = 1, 2$$

$$a_i^* = \delta_i A_i^* \qquad\qquad t > \tau_i \qquad i = 1, 2$$

Proof: Let $X_o = (A_{10}, A_{20})$, $X_1 = (A_1^*, A_{20})$, $X_2 = (A_{10}, A_2^*)$, $X_3 = (A_1^*, A_2^*)$ and R_o be the closed rectangle with vertices X_i, $i = 0, 1, 2, 3$. Since the functions $\Pi(A_1, A_{20})$, $\Pi(A_{10}, A_2)$ are concave in A_1, A_2 respectively, it follows by Assumption 3 that:

$$\Pi(X_o) - \Pi(X_1) \le \Pi_1(X_1)(A_{10} - A_1^*) = 0$$

$$\Pi(X_o) - \Pi(X_2) \le \Pi_2(X_2)(A_{20} - A_2^*) = 0 \ .$$

Hence

$$\Pi(X_i) \ge \Pi(X_o) \ , \qquad i = 1, 2, 3 \ .$$

It is, of course, apparent that any optimal policy must remain within R_o.

Since R_o is a (closed bounded) convex set for any $X \in R_o$ we have

$$\Pi(X) \ge \sum_{i=o}^{3} \lambda_i \Pi(X_i) \ge \sum_{i=o}^{3} \lambda_i \Pi(X_o) = \Pi(X_o)$$

where

$$0 \le \lambda_i \le 1 \ , \quad \sum_{i=o}^{3} \lambda_i = 1 \ .$$

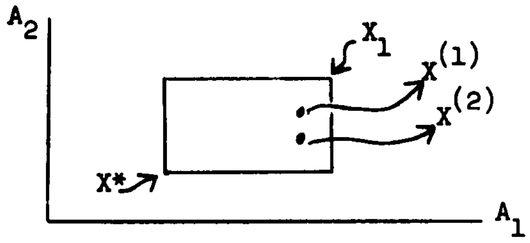
This shows that if the firm maintains none of the stocks, its profit does not decline. We next show that it is not optimal to partially maintain either of the two stocks. Consider any sequence of points, $\{X^{(i)}\}$, such that $X^{(i)} \in R_o$, $X^{(o)} = X_o$, $\lim_{i \to \infty} X^{(i)} = X^*$, and let R_i be the (closed) rectangles having opposite vertices, X^* and $X^{(i)}$. We have $R_o \quad R_1 \quad R_2 \ldots$. An application of the previous argument shows that if X is any point in R_n, then $\prod(X) \geq \prod(X^{(n)})$.

From this it readily follows that partial maintenance of any of the stocks is nonoptimal. Thus optimal policy is to decumulate as fast as possible. Because of the constraints, we must have:

$$A_i(t) = A_{io}\, e^{-\delta_i t} \qquad 0 \leq t \leq \tau_i$$

where $A_i(\tau_i) = A_i^*$, $i = 1, 2$. Thereafter Lemma 2.4 applies.

Perhaps an illustration will clarify the method of reasoning.



Suppose with partial maintenance of A_2, after the lapse of time Δt, we arrive at $X^{(1)}$, in the figure above. Without partial maintenance we arrive at $X^{(2)}$. An application of the previous reasoning shows $\prod(X^{(2)}) \geq \prod(X^{(1)})$. Since it is costly to arrive at $X^{(1)}$ and costless to arrive at $X^{(2)}$, the conclusion is clear.

It remains now to consider the optimal policy when one of the stocks is above and the other below the optimal level. A construction similar to the figure above will show that optimal policy is to increase the deficient stock to its optimal level instantaneously and to let the redundant stock decay at its maximal rate. Details are left for the reader.

3. Extension to the Case of Research Expenditures.

Let A_3 denote the stock of knowledge or level of efficiency which may be thought of as a kind of integral effect of expenditures on research and development.

If assumptions analogous to (3) are made about A_3, then the preceding formulation shows that the lemmas of Section 2 still are valid, provided we make an assumption analogous to Assumption 3 for the function $\Pi(A_1, A_2, A_3, z)$.

The variable A_3 enters into the cost function so that now the firm has two alternatives in manipulating its cost structure. However, the qualitative characterization of the solution will be considerably simplified if we bear in mind that it is very realistic to assume that $\delta_3 = 0$, i.e., that knowledge or efficiency does not decay, or that "we can't forget." Hence if a solution yields $A^*_{30} < A_{30}$, then we ought to treat the level of efficiency as a datum and thus resolve our problem. This is the case where we have friction moving "backward." On the other hand, it is possible to argue that such initial conditions do not or cannot arise in practice so we may neglect this case by suitably restricting the class of admissible initial conditions. In this case, the analogues of Lemmas 2.1, 2.3, and 2.4 are easily established and this, in conjunction with Lemma 2.5, is sufficient to characterize the solution.

REFERENCE

[1] Nerlove, M. and Arrow, K. J., "Optimal Advertising Policy under Dynamic Conditions." *Economica* (forthcoming).

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